2019-Oct-24 (pre-lecture) Prof. Yun William Yu

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Outline:

- Ansätze
- Variation of Parameters
- Reduction of Order
- RLC circuits

Last time:

Theorem Teschl 3.7: Given $x^{(n)} + c_{n-1}x^{(n-1)} + \dots + c_1\dot{x} + c_0 x = 0$, if α_j, $1 \le j \le m$, are the zeros of the characteristic polynomial $z^n + c_{n-1}z^{n-1} + \dots + c_1 z + c_0 = \prod_{j=1}^{m}(z-\alpha_j)^{a_j}$, where a_j are the corresponding multiplicities, then the functions $x_{j,k}(t) = t^k e^{\alpha_j t}$, $0 \le k \le a_j$, $1 \le j \le m$ are n linearly ind. solutions.

Lemma Teschl 3.8: Given $x^{(n)} + c_{n-1}x^{(n-1)} + \dots + c_1\dot{x} + c_0 x = Q(t)$, where $Q(t) = p(t)e^{\beta t}$, where $p(t)$ is a polynomial, then there exists a particular solution of the same form $x_p(t) = q(t)e^{\beta t}$, where $q(t)$ is a polynomial which satisfies $\deg(q) = \deg(p)$ if $\beta \notin \{\alpha_j\}_{j=1}^{m}$ is not equal to any of the zeros of the char. poly and $\deg(q) = \deg(p) + a_j$ if $\beta = \alpha_j$.

This time:

Define: An **ansatz** (plural **ansätze** or **ansatzes**) is an educated guess that is verified later by the results.

Ex. $\dot{x} + c_0 x = 0$ has solution $x = e^{\alpha t}$ for some α.

$\to$ **Ansatz:** $\ddot{x} + c_1\dot{x} + c_0 x = 0$ has solution $x = e^{\alpha t}$ for some α.

Ex. $\left(\frac{d}{dt} - \alpha\right)^a x = 0$ has solution $x = e^{\alpha t}$

$\to$ **Ansatz:** $x = u(t)e^{\alpha t}$ is also a solution (and when we solve, we get that $u(t)$ is a poly of deg $a-1$)

Ex: $\dot{x} - \alpha x = e^{\alpha t}$ can be solved using the IF $e^{-\alpha t}$ (first order linear ODE)

$$\Rightarrow e^{-\alpha t}dx + (-\alpha x e^{-\alpha t} - 1)dt = 0$$

$$\Rightarrow x e^{-\alpha t} - t = C \Rightarrow x = (C + t)e^{\alpha t} = \overbrace{Ce^{\alpha t}}^{x_c} + \overbrace{te^{\alpha t}}^{x_p}$$

$\to$ **Ansatz:** $\left(\frac{d}{dt} - \alpha\right)^a x = t^b e^{\alpha t}$ has a solution that contains $t^{a+b}e^{\alpha t}$.

Method of Variation of Parameters

Given $f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$,
where $f_i(x)$ is continuous and $f_n(x) \neq 0$ for any x on the interval,
if we can find n linearly independent solutions $y_1, \ldots, y_n$ to the homog. eqn.

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0,$$

then we can find a particular solution y_p of the form

$$y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x),$$ where $u_i(x)$ are unknown functions. (ansatz)

We can then solve for $u_i(x)$ by plugging y_p back into the original ODE, which will give the system of equations

$$\begin{aligned}
u_1'y_1 + \cdots + u_n'y_n &= 0\\
u_1'y_1' + \cdots + u_n'y_n' &= 0\\
&\vdots\\
u_1'y_1^{(n-2)} + \cdots + u_n'y_n^{(n-2)} &= 0\\
u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)} &= \frac{Q(x)}{f_n(x)}.
\end{aligned}$$

We can then solve for each u_i', and then integrate to get u_i.

Proof for 2nd order constant coeff.

$a_2y'' + a_1y' + a_0y = Q(x)$, soln y_1, y_2

Guess $y_p = u_1y_1 + u_2y_2$

$$\begin{aligned}
y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'\\
&= (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2)
\end{aligned}$$

$$y_p'' = (u_1y_1'' + u_2y_2'') + (u_1'y_1' + u_2'y_2') + (u_1'y_1 + u_2'y_2)'$$

Then
$$u_1(a_2y_1'' + a_1y_1' + a_0y_1) + u_2(a_2y_2'' + a_1y_1' + a_0y_2) + a_2(u_1'y_1' + u_2'y_2') + a_2(u_1'y_1 + u_2'y_2)' + a_1(u_1'y_1 + u_2'y_2) = Q(x)$$

$$\Rightarrow \left.\begin{aligned} u_1'y_1 + u_2'y_2 &= 0\\ u_1'y_1' + u_2'y_2' &= Q(x)/a_2 \end{aligned}\right\}$$ One solution

Note: This method works for non-constant coefficients and when $Q(x)$ has infinitely many linearly ind. derivatives, but it can be harder to work with than undetermined coefficients.

Ex. $y'' - 3y' + 2y = \sin e^{-x}$

What if we try method of undetermined coefficients?

$$\left.\begin{aligned}
Q(x) &= \sin e^{-x}\\
Q'(x) &= -e^{-x}\cos e^{-x}\\
Q''(x) &= -e^{-2x}\sin e^{-x} + \cdots\\
Q'''(x) &= e^{-3x}\cos e^{-x} + \cdots\\
&\vdots
\end{aligned}\right\}$$ linearly ind., so we can't use method of undetermined coefficients

Let's use variation of parameters insted.

First need 2 ind. solutions to the homogeneous eqn

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$$y''-3y'+2y=0.$$

Char eq. $m^2-3m+2=0 \Rightarrow m=1,2,$

so e^x, e^{2x} are linearly ind. soln. to the homogeneous eqn

Then $y_p = u_1 e^x + u_2 e^{2x}$, and

$$\begin{cases} u_1' e^x + u_2' e^{2x} = 0 \\ u_1' e^x + 2u_2' e^{2x} = \sin e^{-x} \end{cases}$$

$\Rightarrow u_1' e^x = -u_2' e^{2x}, \quad u_1' = -u_2' e^x$

$\Rightarrow u_2' e^{2x} = \sin e^{-x}$

$u_2' = e^{-2x} \sin e^{-x}, \quad u_1' = -e^{-x} \sin e^{-x}$

Integrate letting $u = e^{-x}$, $du = -e^{-x} dx$, $dx = -e^x du = -\frac{du}{u}$

$$\frac{du_2}{dx} = e^{-2x} \sin e^{-x} \qquad \frac{du_1}{dx} = -e^{-x} \sin e^{-x}$$

$$du_2 = e^{-2x} \sin e^{-x}\, dx \qquad du_1 = -e^{-x} \sin e^{-x}\, dx$$

$$du_2 = -u \sin u\, du \qquad du_1 = \sin u\, du$$

Integration by parts

$$u_2 = -\sin u + u \cos u \qquad u_1 = -\cos u$$

$$u_2 = -\sin e^{-x} + e^{-x} \cos e^{-x} \qquad u_2 = -\cos e^{-x}$$

$$\Rightarrow y_p = (-\cos e^{-x}) e^x + (-\sin e^{-x} + e^{-x} \cos e^{-x}) e^{2x}$$
$$= -e^{2x} \sin e^{-x}.$$

Thus, $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x}.$ $\square$

Reduction of Order:

Given a general nth order linear ODE,

$$f_n(x) y^{(n)} + \cdots + f_1(x) y' + f_0(x) y = Q(x)$$

we cannot in general solve it.

However, if we know $n-1$ linearly ind. solutions, we can find one

However, if we know $n-1$ linearly ind. solutions, we can find one additional linearly ind solution using reduction of order.

Only consider 2nd order case

Assume y_1 is a nontrivial solution to

$$f_2(x)y'' + f_1(x)y' + f_0(x)y = 0$$

Ansatz: $y_2(x) = y_1(x)\int u(x)\,dx$, is another lin ind. soln, where $u(x)$ is an unknown fkt.

Verification: $y_2' = y_1 u + y_1' \int u(x)\,dx$

$$y_2'' = y_1 u' + y_1' u + y_1' u + y_1'' \int u(x)\,dx = y_1 u' + 2y_1' u + y_1'' \int u(x)\,dx$$

$$f_2\left[y_1 u' + 2y_1' u + y_1'' \int u(x)\,dx\right] + f_1\left[y_1 u + y_1' \int u(x)\,dx\right] + f_0\left[y_1 \int u(x)\,dx\right] = 0$$

$$\underbrace{\left(f_2 y_1'' + f_1 y_1' + f_0 y_1\right)}_{=0 \text{ because } y_1 \text{ is a solution}} \int u(x)\,dx + \left[2f_2 y_1' + f_1 y_1\right]u + f_2 y_1 u' = 0$$

$$\Rightarrow \left[2f_2 y_1' + f_1 y_1\right]u + f_2 y_1 u' = 0$$

$$u \cdot 2f_2 \cdot \frac{dy_1}{dx} + f_1 y_1 u + f_2 y_1 \cdot \frac{du}{dx} = 0$$

$$u \cdot 2f_2 \cdot dy_1 + u f_1 y_1\,dx + f_2 y_1\,du = 0$$

$$2 \cdot \frac{dy_1}{y_1} + \frac{f_1}{f_2}\,dx + \frac{du}{u} = 0$$

$$2\log y_1 + \int \frac{f_1}{f_2}\,dx + \log u = 0$$

$$\log(u y_1^2) = -\int \frac{f_1}{f_2}\,dx$$

$$u = \frac{\exp\left(-\int \frac{f_1(x)}{f_2(x)}\,dx\right)}{y_1^2}$$

$$\Rightarrow y_2 = y_1 \cdot \int \frac{\exp\left(-\int \frac{f_1(x)}{f_2(x)}\,dx\right)}{y_1^2}\,dx.$$

This y_2 is linearly ind. of y_1, though we won't prove it here.

Ex. $x^2 y'' + xy' - y = 0$, $x \neq 0$, and given $y_1 = x$ is a solution.

$$Y_2 = x \cdot \int \frac{\exp\left(-\int \frac{x}{x^2}dx\right)}{x^2} dx = x \cdot \int \frac{1}{x^3} dx = x \cdot -\frac{1}{2x^2} = -\frac{1}{2x}.$$

or we can rederive from

ansatz:

$$\left.\begin{aligned} y_2 &= x \int u(x)\,dx \\ y_2' &= xu + \int u(x)\,dx \\ y_2'' &= xu' + u + u = xu' + 2u \end{aligned}\right\} \quad x^3u' + 2x^2u + x^2u + x\int u(x)dx - x\int u(x)dx = 0$$

$$x^3u' + 2x^2u + x^2u = 0$$

$$xu' + 2u + u = 0$$

$$xu' + 3u = 0$$

$$x\frac{du}{dx} = -3u \Rightarrow \frac{du}{u} = -3 \cdot \frac{dx}{x}$$

$$\ln u = -3 \ln x = \ln x^{-3}$$

$$u = x^{-3}$$

$$Y_2 = x \int x^{-3} dy$$

$$= x \cdot \frac{1}{2x^2} = \frac{1}{2x}$$

The same substitution also gives a particular solution to the nonhomogeneous case.

Next time: RLC circuits